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THE LAMINAR BOUNDARY LAYER WITH ARBITRARILY DISTRIBUTED MASS TRANSFER*

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Abstract—An analysis is carried out of the velocity, energy and concentration fields in laminar boundary layers with arbitrarily distributed mass transfer at the surface. The approach is based on treating the effect of the mass transfer as a perturbation. The first-order effects are simple to evaluate numerically; higher order effects may be obtained systematically but require modest computing effort for their numerical evaluation. Application is made to the boundary layer on a cone with uniform mass transfer and with an energy and concentration balance imposed at the surface.

NOMENCLATURE

A_n, arbitrary constants, cf. equations (15) and (18);
B_n, arbitrary constants, cf. equation (44);
c_f, skin-friction coefficient;
C_n, square of the norms of the N_n functions, cf. equation (17);
D_n, square of the norms of the M_n functions, cf. equation (46);
f, modified stream function, cf. equation (2);
G, Green's function, cf. equations (23) and (51);
g, stagnation enthalpy ratio, $h_s/h_{s,e}$;
$\hat{g}$, wall enthalpy parameter, cf. equation (64);
h_s, stagnation enthalpy;
j, index, 0 for two-dimensional flow, 1 for axisymmetrical flow;
L, partial differential operator, cf. equations (9) and (43); length of permeable surface;
M_n, eigenfunctions, cf. equation (45);
N_n, eigenfunctions, cf. equation (16);
r, cylindrical coordinate of the body surface;
s, transformed streamwise variable, cf. equation (1);
u, velocity component in the streamwise direction;
v, velocity component in the normal direction;
x, streamwise coordinate;
y, normal coordinate.

Greek symbols

γ_n, eigenvalues, cf. equation (45);
ϵ, mass transfer parameter, cf. equation (26);
η, transformed normal variable, cf. equation (1);
$\varkappa$, wall species parameter, cf. equation (64);
λ_n, eigenvalues, cf. equation (16);
μ, viscosity coefficient;
ξ, dummy variable;
ρ, mass density;
χ, transformed streamwise variable, $\chi \equiv \hat{s}^{\frac{1}{6}} = -f_w$.

Subscripts

e, refers to conditions in the external flow;
i, refers to initial conditions (at $x = s = 0$);

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L, refers to length of permeable surface;
w, refers to conditions at the body surface (wall).

INTRODUCTION

THERE ARE a variety of problems in laminar boundary layer theory and practice which involve mass transfer from and into the surface on which the layer is developing. As a result there exists considerable literature on the effects of mass transfer with and without concomitant species diffusion and energy transfer. Because of the considerable simplifications which accrue, most studies have been concerned with distributions of mass transfer and of properties in the external stream such that similar flow prevails. There are, however, cases for which the requirements of similarity are unacceptable; examples of these are: transient, time-dependent sublimation; the growth of a deposited layer on a cryogenic surface; localized mass transfer such as occurs on surfaces consisting of combinations of permeable and impermeable materials; and porous surfaces with uniform mass transfer and uniform external streams. Relatively little effort appears to have been devoted to these non-similar flows.

It is the purpose of the present paper to develop a method of analysis for flows with uniform external streams but with arbitrarily distributed mass transfer. As will be shown below the method of solution involves successive approximations which are formally identical but which become numerically more complex as the higher approximations are calculated. The first-order effects of mass transfer on the velocity field can be computed by slide rule or desk computer while the same effects on the energy and species fields and the higher order effects on all three involve in general a double quadrature of a complicated integrand and thus require a modest computer. Accordingly, some might consider the present analysis of practical interest only for small but arbitrarily distributed rates of mass transfer.

The basis for the method resides in the perturbation analysis of Libby and Fox [1, 2]. The eigenvalues and eigenfunctions developed there and in [3], supplemented by a few functions which can be computed once and for all, lead to closed-form solutions for first-order effects.

The paper is organized as follows: The solution for the velocity field is presented first and then compared with more accurate results in several typical cases. The related treatment of the energy and species conservation is developed next. The solutions are applied in a numerical example to the flow about a cone with the uniform injection of a foreign, nonreactive gas and with energy and mass balance taken into account. The relevant literature will be referenced as the analysis is developed.

THE VELOCITY FIELD

We consider a laminar boundary layer with a uniform external stream in either a two-dimensional or axisymmetric geometry. We assume that the frequently employed assumption $\rho\mu \simeq \rho_e\mu_e$ is sufficiently accurate. Accordingly, in terms of the modified stream function $f(s, \eta)$ the velocity distribution is given by the partial differential equation (cf. Lees [4] and Hayes and Probstein [5])

$$f_{\eta\eta\eta} + ff_{\eta\eta} - 2s(f_\eta f_{s\eta} - f_s f_{\eta\eta}) = 0 \tag{1}$$

where

$$\eta = \rho_e u_e r^j (2s)^{-\frac{1}{2}} \int_0^y (\rho/\rho_e)\, dy,$$

$$s = \rho_e \mu_e u_e \int_0^x r^{2j}\, dx$$

and where the usual notation is employed. The velocity distribution is given in terms of $f(s, \eta)$ by

$$u/u_e = f_\eta \tag{2}$$

and

$$(v/u_e)[\rho/\rho_e\mu_e r^j] = -(2s)^{\frac{1}{2}}\,[(f/2s) + f_s + (\partial\eta/\partial x) f_\eta]. \tag{3}$$

At the surface equation (3) may be written in a form convenient for later requirements, i.e. as

$$(\rho v)_w/(\rho_e u_e \mu_e r^j) = -\frac{d}{ds}[(2s)^{\frac{1}{2}} f_w]. \quad (4)$$

The initial and boundary conditions to be imposed on the solution of equation (1) for present purposes are

$$\left.\begin{aligned} f(0,\eta) &= f_i(\eta) \\ f_\eta(s,\infty) &= 1 \\ f_\eta(s,0) &= 0 \\ f(s,0) &= f_w(s), \quad \text{given.} \end{aligned}\right\} \quad (5)$$

The initial profile expressed in terms of $f_i(\eta)$ is given by the ordinary differential equation derivable from equation (1) by letting $s \to 0$, namely,

$$f_i''' + f_i f_i'' = 0 \quad (6)$$

subject to $f_i(0) = f_w(0)$, given, $f_i'(0) = 0$, $f_i'(\infty) = 1$; where $(\ )'$ denotes differentiation with respect to η.

We consider a solution to the problem posed above in the form

$$f(s,\eta) = f_0(\eta) + f_1(s,\eta) \quad (7)$$

where f_0 is the Blasius function defined by

$$f_0''' + f_0 f_0'' = 0 \quad (8)$$

subject to $f_0(0) = f_0'(0) = 0$, $f_0'(\infty) = 1$. Then equation (1) can be written in the convenient form

$$L_1 f_1 = (f_1)_{\eta\eta\eta} + f_0(f_1)_{\eta\eta} + f_0'' f_1 - 2s[f_0'(f_1)_{s\eta} - f_0''(f_1)_s]$$
$$= -f_1(f_1)_{\eta\eta} + 2s[(f_1)_\eta(f_1)_{s\eta} - (f_1)_s(f_1)_{\eta\eta}]. \quad (9)$$

Correspondingly, equation (6) can be written as

$$f_{1,i}''' + f_0 f_{1,i}''' + f_0'' f_{1,i} = -f_{1,i} f_{1,i}'' \quad (10)$$

where $f_{1,i}(\eta) \equiv f_1(0,\eta)$.

Before considering the boundary conditions applicable to $f_1(s,\eta)$ we expose our method of solution which may be considered as one based on perturbation techniques. Suppose, for instance, we put an iteration index on f_1 in equation (9) and on $f_{1,i}$ in equation (10) such that on the left-hand sides we have $f^{(k)}$ and $f_{1,i}^{(k)}$ and on the right-hand sides we have $f_1^{(k-1)}$ and $f_{1,i}^{(k-1)}$. The solution obtained for $f_1^{(1)}$ with $f_1^{(0)} \equiv 0$ corresponds to a first-order perturbation solution. However, the way will be open for successive improvements, i.e. for $k \geqslant 2$; these, of course, correspond to higher order perturbation solutions. If, as we shall largely do here, we actually compute only the first-order solutions, the mass transfer expressed in terms of $f_w(s)$ is implied small in absolute value compared with unity.

We shall satisfy the boundary conditions with the approximation corresponding to $k = 1$; accordingly, these conditions for $k \geqslant 2$ will be homogeneous. Thus consider the problem for $k = 1$ with again $f_1^{(0)} \equiv 0$; we have

$$L_1 f_1^{(1)} \equiv (f_1^{(1)})_{\eta\eta\eta} + f_0(f_1^{(1)})_{\eta\eta} + f_0'' f_1^{(1)}$$
$$- 2s[f_0'(f_1^{(1)})_{s\eta} - f_0''(f_1)_s] = 0 \quad (11)$$

subject to

$$\begin{aligned} f_1^{(1)}(0,\eta) &= f_{1,i}^{(1)}(\eta) \\ f_1^{(1)}(s,0) &= f_w(s), \quad \text{given} \\ (f_1^{(1)})_\eta(s,0) &= (f_1^{(1)})_\eta(s,\infty) = 0 \end{aligned}$$

and

$$(f_{1,i}^{(1)})''' + f_0(f_{1,i}^{(1)})'' + f_0'' f_{1,i}^{(1)} = 0 \quad (12)$$

subject to

$$f_{1,i}^{(1)} = f_w(0), \quad \text{given};$$
$$(f_{1,i}^{(1)})'(0) = (f_{1,i}^{(1)})'(\infty) = 0.$$

The analysis of Libby and Fox [1] provides a means for solving the problem posed by equations (11) and (12); consider first the solution for the initial profile, i.e. the solution of equation (12). This may be expressed directly in terms of the function denoted previously as $f_2^{(1)}$, a function with the boundary conditions $f_2^{(1)}(0) = 1$, $(f_2^{(1)})'(0) = (f_2^{(1)})'(\infty) = 0$. It is unnecessary for present purposes to carry the superscript unity so we denote this function by $f_2(\eta)$; for completeness $f_2'(\eta)$ and the crucial wall value $f_2''(0)$ are shown in Fig. 1. Thus

$$f_{1,i}^{(1)}(\eta) = f_w(0)f_2(\eta). \qquad (13)$$

The solution for $f_1^{(1)}(s, \eta)$ can be built up from a unit solution $\tilde{f}(s, \eta; \xi)$ defined by

$$L_1\tilde{f} = 0 \qquad (14)$$

subject to

$$\tilde{f}(0, \eta; \xi) = 0$$

$$\tilde{f}_\eta(s, 0; \xi) = \tilde{f}_\eta(s, \infty; \xi) = 0$$

$$\tilde{f}(s, 0; \xi) = 0, \qquad 0 < s < \xi$$

$$= 1, \qquad \xi < s.$$

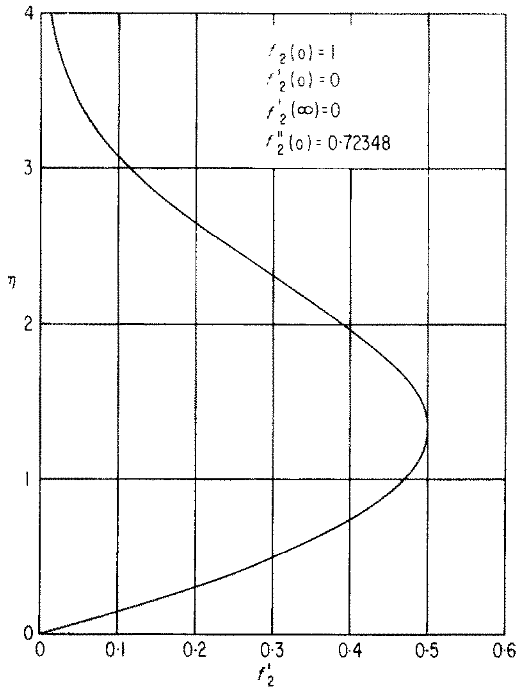

FIG. 1. Velocity profile for unit mass transfer.

The solution to this problem within some arbitrary constants A_n is

$$\tilde{f}(s, \eta; \xi) \equiv 0, \qquad 0 < s < \xi$$

$$= f_2(\eta) + \sum_{n=1}^{\infty} A_n(s/\xi)^{-\frac{1}{2}\lambda_{1,n}} N_{1,n}(\eta), \qquad \xi < s \qquad (15)$$

where the $\lambda_{1,n}$ and related $N_{1,n}(\eta)$ are the eigenvalues and eigenfunctions given by Libby and Fox [1] and with greater numerical accuracy by Libby [3]. Mathematically, they are defined by the ordinary differential equation

$$N_n''' + f_0 N_n'' + \lambda_n f_0' N_n' + (1 - \lambda_n) f_0'' N_n = 0 \quad (16)$$

subject to the conditions $N_n(0) = N_n'(0) = N_n'(\infty) = 0$ and by the requirement that as $\eta \to \infty$, $N_n' \sim \exp[-\frac{1}{2}(\eta - \varkappa)^2]$, i.e. that the power function decay, which exists in the asymptotic solution of equation (16), is suppressed. In previous work it was shown that the eigenfunctions so defined form a complete orthogonal set for functions whose derivative vanishes as $\eta \to \infty$ at least as fast as $\exp[-\frac{1}{4}(\eta - \varkappa)^2]$. The orthogonality condition is expressed as

$$\int_0^\infty [(f_0')^4/f_0''] (N_n/f_0')' (N_m/f_0')' \, d\eta = C_n \delta_{mn}. \quad (17)$$

In previous work [3] the first 20 eigenvalues and the square of their norms, C_n, for $N_n''(0) = 1$ have been presented.

The A_n coefficients in equation (15) must be determined so that $\tilde{f}(s, \eta; s) = 0, \eta > 0$. Although there are a variety of means for employing this condition for determining the A_n coefficients, we choose to exploit the orthogonality property of the N_n functions so that this condition is satisfied in an integral sense; we obtain from equation (17)

$$A_n = -C_n^{-1} \int_0^\infty [(f_0')^4/f_0''] (N_n/f_0')' (f_2/f_0')' \, d\eta. \quad (18)$$

The values of A_n and the first 20 eigenvalues are given in Table 1.

Table 1. The eigenvalues and coefficients for the velocity solution

n	λ_n	A_n	n	λ_n	A_n
1	2	0·2205	11	20·979	0·157
2	3·774	0·2309	12	22·920	0·153
3	5·629	0·2179	13	24·865	0·149
4	7·513	0·2057	14	26·811	0·146
5	9·414	0·1948	15	28·760	0·142
6	11·327	0·1859	16	30·710	0·140
7	13·247	0·1784	17	32·662	0·137
8	15·173	0·1719	18	34·615	0·135
9	17·104	0·1661	19	36·570	0·132
10	19·040	0·1610	20	38·526	0·130

With the solutions for $f_{1,i}^{(1)}(\eta)$ and $\tilde{f}(s, \eta; \xi)$ we can build up the solution for $f_1^{(1)}(s, \eta)$ by means of a Duhamel integral; we get

$$f_1^{(1)}(s, \eta) = f_w(0) f_2(\eta) + \int_0^s \tilde{f}(s, \eta; \xi) \left(\frac{d f_w}{d\xi}\right) d\xi. \quad (19)$$

Substitution of the solution for $\tilde{f}(s, \eta; \xi)$ and integration by parts leads after some rearrangement to a convenient form for the desired solution, namely,

$$f_1^{(1)}(s, \eta) = f_w(s)\left[f_2(\eta) + \sum_{n=1}^\infty A_n N_n(\eta)\right] - \sum_{n=1}^\infty (A_n \lambda_n/2) N_n(\eta) s^{-\frac{1}{2}\lambda_n} \int_0^s \xi^{\frac{1}{2}\lambda_n - 1} f_w(\xi) \, d\xi. \quad (20)$$

Although we will not compute the higher order solutions, it is perhaps of interest to point out that a formal solution for them can be written down. The problem for $f_1^{(2)}(s, \eta)$ is found from equations (9) and (11) to be

$$L_1 f_1^{(2)} = H^{(2)}(s, \eta) = -f_1^{(1)} (f_1^{(1)})_{\eta\eta} + 2s[(f_1^{(1)})_\eta (f_1)_{s\eta}^{(1)} - (f_1^{(1)})_s (f_1^{(1)})_{\eta\eta}] \quad (21)$$

subject to the conditions

$$f_1^{(2)}(0, \eta) = f_{1,i}^{(2)}(\eta)$$

$$f_1^{(2)}(s, 0) = (f_1^{(2)})_\eta(s, 0) = (f_1^{(2)})_\eta(s, \infty) = 0$$

where the next order solution for the initial profile, $f_{1,i}^{(2)}(\eta)$, is given from equations (10) and (13) as

$$(f_{1,i}^{(2)})''' + f_0 (f_{1,i}^{(2)})'' + f_0'' f_{1,i}^{(2)} = -f_w^{(2)}(0) f_2(\eta) f_2''(\eta) \quad (22)$$

subject to

$$f_{1,i}^{(2)}(0) = (f_{1,i}^{(2)})_\eta(0) = (f_{1,i}^{(2)})_\eta(\infty) = 0.$$

The solution for equation (22) can be obtained readily either by numerical integration or by quadrature using the independent solutions to the homogeneous equation given by Libby and Fox [1]. Moreover, the solution of equation (21) may be obtained in terms of the Green's function $G_1(s, \eta; s_0, \eta_0)$ given previously [1]; there results

$$f_1^{(2)}(s, \eta) = f_{1,i}^{(2)}(\eta) + \int_0^\infty \int_0^s G_1(s, \eta; s_0, \eta_0) \times H^{(2)}(s_0, \eta_0) \, ds_0 \, d\eta_0. \quad (23)$$

The numerical evaluation of the double integral here requires a modest computer; for some relatively simple distributions $f_w(s)$ the double integral becomes a series of single integrals. Clearly, higher order perturbations can be determined.

COMPARISON OF THE VELOCITY FIELD WITH OTHER ANALYSES

The first-order solution given by equations (7) and (20) can be applied to several problems for which more accurate solutions are available. Emmons and Leigh [6] provide solutions for injection distributed so as to correspond to similar flow, i.e. for f_w negative and constant. In this case our first-order solution is

$$f(s, \eta) = f(\eta) \simeq f_0(\eta) - (-f_w)f_2(\eta) \quad (24)$$

so that the gross quantity of technical interest, f''_w is given by the linear relation

$$f''_w/f''_{0,w} \simeq 1 - 1{\cdot}54\,(-f_w). \quad (25)$$

Figure 2 shows the comparison between equation (25) and the exact numerical results of Emmons and Leigh put in terms of our variables; excellent agreement to $-f_w \simeq 0{\cdot}4$ will be noted.

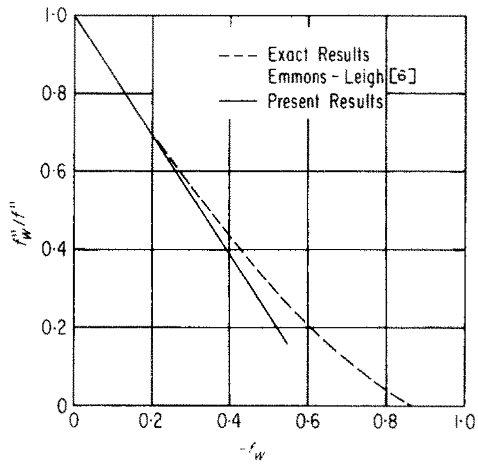

FIG. 2. Comparison of predicted effect of injection on skin friction.

The case of the boundary layer with uniform mass transfer, i.e. of $(\rho v)_w$ = constant, has been treated by several authors. Most recently Smith and Clutter [7] have applied their difference–differential technique to the case of uniform suction for constant properties flow while Libby and Chen [8] have obtained for the compressible case a series solution which is in terms of a mass transfer parameter $\epsilon \equiv [-(\rho v)_w(s/2)^{\frac{1}{2}}/(\rho_e u_e \mu_e)]$, which is valid for either suction or injection, and which is carried out by them to five terms. If $(\rho v)_w$ is constant everywhere, equation (4) yields the result that

$$f_w(s) = \epsilon(s) \sim s^{\frac{1}{2}} \quad (26)$$

and equation (20) becomes

$$f(s, \eta) \simeq f_0(\eta) + \epsilon(s)[f_2(\eta) + \sum_{n=1}^{\infty} A_n(1 + \lambda_n)^{-1} N_n(\eta)]. \quad (27)$$

The function of η within [ ] of equation (27) corresponds to the function $N_1(\eta)$ in the notation of Libby and Chen. We note here that the series in n is rather slowly convergent, the $A_n(1 + \lambda_n)^{-1}$ sequence decreasing by a decade in 10 terms and by a factor of 20 in 20 terms. This and similar behavior in other cases prompted the author to compute additional values of λ_n and C_n and thus of A_n [3]. A measure of the accuracy obtained from the available terms may be obtained by considering the derivative at the wall, namely

$$f_{\eta\eta}(s, 0) = f''_w = 0{\cdot}46960 + \begin{Bmatrix} 0{\cdot}9765 \\ 1{\cdot}025 \end{Bmatrix} \epsilon \quad (28)$$

where the upper number is obtained from 10 terms and the lower from 20. This coefficient may be compared to the value 1·225 from Libby and Chen. In view of the slow convergence of the sequence of partial sums the difference in these two coefficients is not unreasonable. The results given by equation (28) are shown in Fig. 3 along with those obtained by Libby and Chen; it will be noted that our present first-order results would appear to be entirely sufficient for most purposes for injection, i.e. for $\epsilon < 0$ but to be of limited accuracy for suction. In cases of suction for $f_w \gtrsim 0{\cdot}2$ higher order approximations are required.

We consider next the problem treated by Pallone [9] and by Howe [10]; this involves a permeable surface with injection distributed so that a similar flow (f_w = negative constant) prevails over a length of two-dimensional surface L. Downstream of this length the surface is impermeable. In this case the present analysis

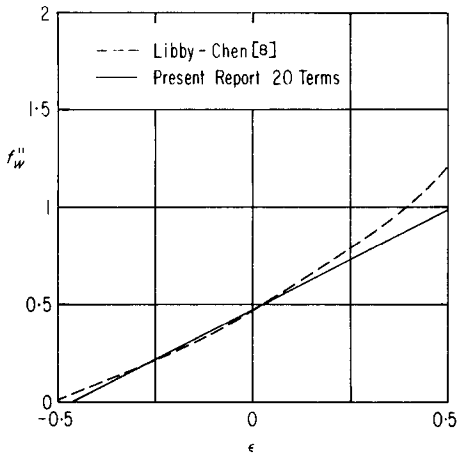

FIG. 3. Comparison of predicted shear parameter with uniform mass transfer.

would provide for the initial section $0 \leqslant s < s_L$, where $s_L = \rho_e \mu_e u_e L$, the solution given by equation (24) with $f_w = f_w(0)$; for $s > s_L$ equation (4) yields

$$f_w(s) = f_w(0)(s_L/s)^{\frac{1}{2}} \tag{29}$$

so that equation (20) becomes in this case, again for $s > s_L$

and for $f_w(0) = -0{\cdot}354$ is compared in Fig. 4 with the prediction of Pallone [9];* the agreement may be seen to be satisfactory. Note also that Libby and Fox [1] carried out a similar first-order calculation from an entirely different point of view and indeed gave second-order values at several values of x/L. The present results are in good agreement with those previously obtained and with more accurate calculations.

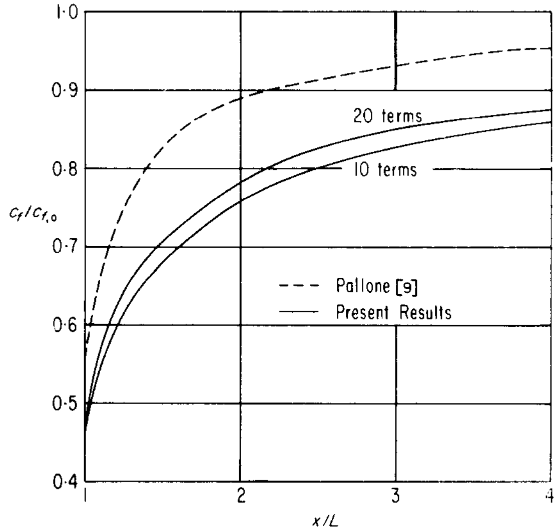

FIG. 4. Comparison of predicted distribution of skin-friction downstream of permeable wall with $-f_w(0) = 0{\cdot}354$.

$$f(s,\eta) = f_0(\eta) + f_w(0)(s/s_L)^{-\frac{1}{2}} \{[f_2(\eta) + \sum_{n=1}^{\infty} A_n N_n(\eta)] - \sum_{n=1}^{\infty} A_n N_n(\eta)[\lambda_n - (s/s_L)^{-\frac{1}{2}(\lambda_n - 1)}](\lambda_n - 1)^{-1}\}. \tag{30}$$

Note of course that $s/s_L = x/L$.

The quantity which is usually employed in the comparison of various solutions to this problem is the distribution of the ratio of skin-friction coefficient with upstream injection to that which would prevail at the same station without injection; i.e. $c_f/c_{f,0} = f_{\eta\eta}(s,0)/f''_{0,w}$. It is perhaps of interest to note that the x-wise derivative of this skin-friction ratio at $x/L = 1$ is finite. The distribution of this ratio with x/L for $x/L \geqslant 1$

We conclude from these comparisons that provided $-0{\cdot}5 \lesssim f_w \lesssim 0{\cdot}2$, the present analysis provides a rapid and reasonably accurate solution. Clearly it can be readily applied to cases

* It should be noted here that the initial profile data used by Pallone came from Low [11] whose definition of f differs from ours by $2^{\frac{1}{2}}$. The identification of $f_w(0) = -0{\cdot}5$ in Libby and Fox [1] for their comparison is ambiguous although the numerical results are correct.

not previously treated, e.g. to the Pallone problem for axisymmetric flow and for uniform injection over the section $0 \leqslant x < L$, and to cases of partial suction for compressible and/or axisymmetric flow with some assurance that reasonably accurate predictions will result. These first-order results can be obtained by use of desk computers and even by slide rule. Somewhat more ambitious computations involving equation (23) will provide more accurate results and will extend the range of f_w which can be accurately considered.

SOLUTION OF THE RELATED ENERGY AND SPECIES CONSERVATION EQUATION

In many of the flow problems which involve mass transfer and which have velocity fields describable by the solution presented above, the related heat transfer is of interest. Indeed in cases of injection the gases introduced by the mass transfer are frequently different from those in the external stream so that both the energy and species fields are frequently desired. The previous work of Fox and Libby [2] and of Fox [12] has provided a basis for the appropriate solution of the equations for conservation of the energy and species according to either an iteration or a perturbation point of view. Applications involving mass transfer in the past have been to special cases, e.g. to the Pallone problem. Accordingly, it appears of interest to develop here the solutions related to the velocity field with arbitrary mass transfer presented above. We note that flows involving gas-phase chemical reaction require an iterative procedure for obtaining the solution of each order (cf. Fox [12]) and will not be considered here.

We adopt the frequently employed approximation of simplified transport properties, $\rho\mu \simeq$ constant, and the Prandtl and Schmidt numbers equal to unity* and consider first the energy equation. Under the present assumptions the energy equation in terms of the stagnation enthalpy ratio, $g \equiv h_s/h_{s,e}$, is

$$g_{\eta\eta} + fg_\eta - 2s(f_\eta g_s - f_s g_\eta) = 0 \tag{31}$$

subject to the following conditions which will be discussed below:

$$g(0, \eta) = g_i(\eta)$$
$$g(s, \infty) = 1$$
$$g(s, 0) = g_w(s).$$

The initial distribution $g_i(\eta)$ is obtained from the solution of

$$g_i'' + f_i g_i' = 0 \tag{32}$$

subject to

$$g_i(0) = g_w(0)$$
$$g_i(\infty) = 1.$$

The boundary condition at the surface has been expressed in terms of a distribution of g_w; there are problems for which this is immediately the proper condition, but there also arise problems for which the proper surface condition involves some relation among $g_w(s)$, $g_\eta(s, 0)$ and $f_w(s)$*. In these problems the solution is conveniently found by making an *a priori* assumption that $g_w(s)$, given but arbitrary, is known and by forming an integral equation implied by the proper boundary condition. We thus adopt the view at the moment that $g_w(s)$ is known.

With equation (7) substituted into equation (31) the latter can be conveniently written as

$$L_2 g = g_{\eta\eta} + f_0 g_\eta - 2sf'_0 g_s = -f_1 g_\eta + 2s[(f_1)_\eta g_s - (f_1)_s g_\eta]. \tag{33}$$

Similarly equation (32) becomes

$$g_i'' + f_0 g_i' = -f_{1,i} g_i'. \tag{34}$$

We now consider a perturbation point of view whereby

$$g(s, \eta) \simeq g^{(1)}(s, \eta) + g^{(2)}(s, \eta) + \dots$$
$$g_i(\eta) = g_i^{(1)}(\eta) + g_i^{(2)}(\eta) + \dots \tag{35}$$

* It will be recognized that the ordering point of view discussed after equation (10) permits inclusion of the effects of variable transport properties if desired.

* Transient sublimation would be such a case.

so that the first two orders are given by

$$L_2 g^{(1)} = 0 \quad (36)$$

$$L_2 g^{(2)} = -f_1^{(1)} g_\eta^{(1)} + 2s[(f_1^{(1)})_\eta g_s^{(1)} - (f_1^{(1)})_s g_\eta^{(1)}] \quad (37)$$

$$(g_i^{(1)})'' + f_0 (g_i^{(1)})' = 0 \quad (38)$$

$$(g_i^{(2)})'' + f_0 (g_i^{(2)})' = -f_{1,i}^{(1)} (g_i^{(1)})' \quad (39)$$

and where higher approximations for g can be successively obtained. It is noted here that if a first-order effect of the deviations of the velocity field from that described by the Blasius solution on the energy field is to be computed, then $g^{(2)}(s, \eta)$ must be determined since $g^{(1)}(s, \eta)$ depends only on $f_0(\eta)$.

The solution for the first-order initial profile is

$$g_i^{(1)} = g_w(0) + (1 - g_w(0)) f_0' \quad (40)$$

so that the solution for the second order is given by

$$(g_i^{(2)})'' + f_0 (g_i^{(2)})' = -(1 - g_w(0)) f_0'' (f_{1,i}^{(1)}) \quad (41)$$

subject to homogeneous boundary conditions. The solution of equation (41) is

$$g_i^{(2)} = -(1 - g_w(0))\Big[\int_0^\eta f_0'' \Big(\int_0^\eta f_{1,i}^{(1)}\, d\eta\Big) d\eta - f_0' \int_0^\infty f_0'' \Big(\int_0^\eta f_{1,i}^{(1)}\, d\eta\Big) d\eta\Big]. \quad (42)$$

Similarly, closed-form solutions can be obtained for the higher order approximations for the initial profile. Note that if the flow is such that a similar solution prevails, i.e. that $g_w(s) \equiv g_w(0)$ and $f_w(s) \equiv f_w(0)$, then this initial solution prevails for all s and is the second-order solution for the entire flow.

The solution for $g^{(1)}$ can be constructed from a unit solution in a fashion similar to that used to develop $f_1^{(1)}$ [cf. equations (14) and (15)]; consider a unit solution $\mathscr{g}(s, \eta; \xi)$ such that

$$L_2 \mathscr{g} = 0 \quad (43)$$

$$\mathscr{g}(0, \eta; \xi) = \mathscr{g}(s, \infty; \xi) = 0$$

$$\mathscr{g}(s, 0; \xi) = 0, \qquad 0 \leqslant s < \xi$$
$$= 1, \qquad \xi < s.$$

The solution to equations (43) is

$$\mathscr{g}(s, \eta; \xi) = 0, \qquad 0 \leqslant s < \xi$$
$$= (1 - f_0') = \sum_{n=1}^{\infty} B_n (s/\xi)^{-\frac{1}{2}\gamma_n} M_n(\eta), \qquad \xi < s \quad (44)$$

where the γ_n and the $M_n(\eta)$ are the eigenvalues and eigenfunctions given by Fox and Libby [2] and identified by them as $\lambda_{1,n}$ and $N_{1,n}(\eta)$.* For completeness we state that the $M_n(\eta)$ functions are given by the differential equation

$$M_n'' + f_0 M_n' + \gamma_n f_0' M_n = 0 \quad (45)$$

subject to

$$M_n(0) = M_n(\infty) = 0.$$

Again provided exponential decay of M_n as $\eta \to \infty$ is required, the $M_n(\eta)$ functions form a complete orthogonal set with respect to all functions which decay exponentially to zero as $\eta \to \infty$ and satisfy the orthogonality condition

$$\int_0^\infty (f_0'/f_0'') M_n M_m \, d\eta = D_n \delta_{mn}. \quad (46)$$

In previous work the first 10 eigenvalues, the norms for $M_n'(0) = 1$, and the eigenfunctions have been presented.

The B_n coefficients in equation (44) can be

* The notation has been changed since we need here several different sets of eigenfunctions.

determined so that the function $\bar{g}(s, \eta; \xi)$ is continuous at $s = \xi$ by, e.g., utilizing equation (46). Indeed in this case the integrals can be evaluated analytically so as to obtain

$$B_n = (\gamma_n D_n f''_{0,w})^{-1}. \qquad (47)$$

We list here in Table 2 for completeness the values of γ_n and B_n obtained previously.

Table 2. The eigenvalues and coefficients for the energy and species solutions

n	γ_n	B_n	n	γ_n	B_n
1	1·572	0·1844	6	10·96	0·0597
2	3·385	0·1220	7	12·88	0·0539
3	5·25	0·0923	8	14·81	0·0497
4	7·14	0·0777	9	16·74	0·0462
5	9·05	0·0689	10	18·68	0·0441

Thus the solution for $g^{(1)}$ may now be written down; it is

$$g^{(1)}(s, \eta) = g_w(0) + (1 - g_w(0))f'_0 + \int_0^s \bar{g}(s, \eta; \xi)(\mathrm{d}g_w/\mathrm{d}\xi)\,\mathrm{d}\xi. \qquad (48)$$

After substitution of equation (44) and some rearrangement, there is obtained the convenient form

$$g^{(1)}(s, \eta) = f'_0 + g_w(s)\left[(1 - f'_0) - \sum_{n=1}^{\infty} B_n M_n(\eta)\right] + \sum_{n=1}^{\infty} (B_n\gamma_n/2)\, M_n(\eta)\, s^{-\frac{1}{2}\gamma_n} \int_0^s \xi^{\frac{1}{2}\gamma_n - 1}\, g_w(\xi)\,\mathrm{d}\xi. \qquad (49)$$

Since we shall be interested in $g^{(2)}(s, \eta)$, we consider the solution of equation (37). If the right-hand side thereof is specified symbolically as $J(s, \eta)$, then the solution for $g^{(2)}(s, \eta)$ satisfying the appropriate initial and boundary conditions is expressible in terms of the Green's function $G_2(s, \eta; s_0, \eta_0)$ involving $M_n(\eta)$ functions (cf. Fox and Libby [2]); the result is

$$g^{(2)}(s, \eta) = g_i^{(2)}(\eta) + \int_0^{\infty}\int_0^s G_2(s, \eta; s_0, \eta_0) J(s_0, \eta_0)\,\mathrm{d}s_0\,\mathrm{d}\eta_0, \qquad (50)$$

where in terms of the present notation

$$G_2(s, \eta; s_0, \eta_0) = -\sum_{n=1}^{\infty}\left(\frac{M_n(\eta)\, s^{-\frac{1}{2}\gamma_n}}{2D_n}\right)\left(\frac{M_n(\eta_0)\, s_0^{\frac{1}{2}\gamma_n - 1}}{f''_0(\eta_0)}\right). \qquad (51)$$

With equation (51), and with equations (19) and (48) substituted into the definition of J and the result considered a function of s_0 and η_0, the integrand in equation (50) is explicitly given provided $f_w(s)$ and $g_w(s)$, now interpreted as $f_w(s_0)$ and $g_w(s_0)$ respectively, are known. The numerical evaluation of $g^{(2)}(s, \eta)$ including a number of eigenfunctions in G_2, $f_1^{(1)}$ and $g^{(1)}$ requires a modest computing effort.

We have thus developed the formal solutions for the first- and second-order solution for the energy equation under the assumption that $g_w(s)$ is known. Higher order solutions can be readily generated and lead to forms such as equation (50). We repeat our previous remarks that if $g_w(s)$ is unknown but related in some way to $g_\eta(s, 0)$, then the above solutions may be used to generate an integral equation which defines $g_w(s)$ and which may be solved numerically by iteration. The numerical example discussed below will demonstrate this technique.

There is little which needs to be added to the above discussion to apply it to the analysis of species conservation provided, as we shall assume here, no gas-phase chemical reaction takes place. Let the symbol Y_i denote the mass fraction of species i. Then if, as is frequently done, it is assumed that a single diffusion coefficient exists and that the Lewis number based thereon is unity, then the conservation equation for Y_i is the same as equation (33) with g replaced by Y_i. Furthermore, if we take the point of view that $Y_i(s, 0) \equiv Y_{i,w}(s)$ is given, then the entire analysis for $Y_i^{(1)}$ and $Y_i^{(2)}$ completely parallels that for $g^{(1)}$ and $g^{(2)}$ with the single exception that the value of Y_i as $\eta \to \infty$ in general will be nonunity. Thus the infinity condition on the initial distribution for Y_i, i.e. on $Y_i^{(1)}(0, \eta)$, must be altered. Now just as in the case of the energy equation there are applied problems for which the surface concentration is known *a priori*; an example is a gas mixture flowing over a surface with sufficient catalytic efficiency to maintain an equilibrium concentration at the surface. There are also problems whose solution is given by forming an integral equation for $Y_{i,w}(s)$ from the surface boundary condition and from the solution obtained on the basis of $Y_{i,w}$ being known. Our numerical example discussed below will illustrate this technique.

APPLICATION

As an application of the above analysis to a problem which leads to an integral equation for $g_w(s)$ we consider the boundary layer on a cone with the uniform injection of a foreign, nonreactive gas. We assume that the energy and concentration of the injected gas internal to the porous surface are constants and that the energy transferred to the exposed surface of the cone is all absorbed by the injected gas.* The two-dimensional counterpart of this problem has been done previously by Libby and Chen [8] and their series expansion method suitably modified to account for the behavior of $f_w(s)$ with s, namely $f_w \sim s^{\frac{1}{6}}$ rather than $f_w \sim s^{\frac{1}{2}}$ has recently been redone by Libby [13]. However, since the range of $f_w(s)$ of interest in injection with no external sources of energy transfer is rather limited, the first-order solution obtained here may be sufficiently accurate for most purposes. In addition, of course, the present analysis is readily applied to nonuniform, nonsimilar distribution of mass transfer whereas that of [8] and [13] applies only to $(\rho v)_w$ = constant.

We consider first the velocity field; on a cone $r \sim x \sim s^{\frac{1}{3}}$ so that with $(\rho v)_w$ = constant, equation (4) yields

$$-f_w(s) = \frac{3^{\frac{2}{3}}(\rho v)_w}{\rho_e u_e 2^{\frac{3}{2}} \sin^{\frac{1}{3}}\theta_c}\left(\frac{\rho_e^2 u_e^2 s}{\mu_e^4}\right)^{\frac{1}{6}} \sim s^{\frac{1}{6}} \tag{52}$$

where θ_c is the cone half angle. We shall be able to formulate the solution for the velocity, energy and species fields in terms of $\hat{s}$ where

$$-f_w(\hat{s}) = \hat{s}^{\frac{1}{6}} \tag{53}$$

and where comparison of equations (52) and (53) clearly yields the relation between $\hat{s}$ and s. Thus there must be explictly specified the particular mass transfer rate, cone angle, external flow, etc., only when transformation back to physical x, y variables is performed.

The solution for the velocity distribution is obtained from equation (20) as

$$f(\hat{s}, \eta) = f_0(\eta) - \hat{s}^{\frac{1}{6}}\left[f_2(\eta) + \sum_{n=1}^{\infty} \hat{A}_n N_n(\eta)\right] \tag{54}$$

* We assume here that there are no external sources of energy flux and thus that conduction and diffusion at the surface provide the total energy transfer.

where

$$\hat{A}_n = A_n(1 + 3\lambda_n)^{-1}*.$$

From equation (54) we can compute the entire velocity field on the cone including the skin-friction parameter, $f''_w(\hat{s})$. We note that for $\hat{s}^{\frac{1}{2}} \simeq 0{\cdot}55$ the shear becomes vanishingly small when the 10 terms in the summation are included so we restrict our interest to $\hat{s}^{\frac{1}{2}} \lesssim \frac{1}{2}$. We note also that we do not consider the interesting question of the value of $\hat{s}$ for which $f''_w = 0$, i.e. of the "blow-off" value for this conical case. Such a question can be answered neither by the present analysis nor by that of [8] and [13].

Consider next the solution of the energy equation; in this case the distribution of $g_w(s)$ is not known *a priori* but is to be determined by an energy balance. If the symbol g_c denotes the ratio of the enthalpy of the injected gas to the stagnation enthalpy of the external stream, and if, as mentioned above, the only mechanism for balancing the energy transmitted to the porous surface is a change of the enthalpy of the injected gas, then the boundary condition at the exposed surface is

$$g_\eta(\hat{s}, 0) = \tfrac{4}{3}\hat{s}^{\frac{1}{2}}(g_w - g_c). \tag{55}$$

Our program then will be to find $g^{(1)}$ and $g^{(2)}$ under the assumption that g_w is known and then to formulate an integral equation from equation (55) yielding the unknown distribution of g_w.

We start by noting that equation (55) implies $g_w(0) = 1$ and thus that

$$\begin{aligned} g_i^{(1)} &\equiv 1 \\ g_i^{(2)} &\equiv 0. \end{aligned} \tag{56}$$

Thus we take the point of view that although g_w is arbitrary its value at the origin is taken as its expected value, namely unity. The solution for $g^{(1)}(\hat{s}, \eta)$ is thus given by equation (49); we shall need

$$g_\eta^{(1)}(\hat{s}, 0) = f''_{0,w} - g_w(\hat{s})\,[f''_{0,w} + \sum_{n=1}^{\infty} B_n] + \sum_{n=1}^{\infty} (B_n\gamma_n/2)\,\hat{s}^{-\frac{1}{2}\gamma_n} \int_0^s \xi^{\frac{1}{2}\gamma_n - 1}\, g_w(\xi)\, d\xi. \tag{57}$$

We note that if a solution, which does not account for the first-order alteration of the velocity field, is sufficient, equation (57) used in equation (55) yields an integral equation for $g_w(\hat{s})$ which may be solved numerically.

Consider next $g^{(2)}(\hat{s}, \eta)$; with the first-order solutions given by equations (54) and (57) and with the Green's function given by equation (51), the elements necessary to treat equation (50) with g_w assumed known are available. We are interested in $g_\eta^{(2)}(\hat{s}, 0)$; in this case the double integral in equation (50) can be reduced to a series of single integrals. The results are in a form more convenient for numerical analysis if the dependent variable is changed from g_w to $\hat{g} \equiv (1 - g_w)/(1 - g_c)$ and from $\hat{s}$ to $\chi \equiv \hat{s}^{\frac{1}{2}}$. Thus equation (57) becomes

$$g_\eta^{(1)}(\chi, 0) = (1 - g_c)\{\hat{g}[f''_{0,w} + \sum_{n=1}^{\infty} B_n] - \sum_{n=1}^{\infty} 3\gamma_n B_n\,\chi^{-3\gamma_n} \int_0^\chi \xi^{3\gamma_n - 1}\,\hat{g}\, d\xi\} \tag{58}$$

and after considerable algebra there is obtained

$$\begin{aligned} g_\eta^{(2)}(\chi, 0) = -(1 - g_c) \sum_{n=1}^{\infty} D_n^{-1}\, \{[\chi][U_n\hat{g} + \sum_{j=1}^{\infty} 6V_{nj}\,\chi^{-3\gamma_j} \int_0^\chi \xi^{3\gamma_j - 1}\,\hat{g}\, d\xi] \\ + [6\chi^{-3\gamma_n} \int_0^\chi \xi^{3\gamma_n}\,\hat{g}\, d\xi][(\tfrac{2}{3})Q_n - \langle(3\gamma_n + 1)/6\rangle\, U_n] \\ + [36\chi^{-3\gamma_n} \int_0^\chi s_0^{3\gamma_n - 3\gamma_j} (\int_0^s \xi^{3\gamma_j - 1}\hat{g}\, d\xi)\, ds_0][(\tfrac{2}{3}R_{nj} - \langle(3\gamma_n + 1)/6\rangle V_{nj}]\} \end{aligned} \tag{59}$$

* It is interesting to note that for the two-dimensional case the factor $(1 + 3\lambda_n)$ becomes $(1 + \lambda_n)$ so the velocity field and indeed the solutions for the two cases are different and not scaleable. This is also implied by Libby and Chen [8].

where

$$U_n = \int_0^\infty (M_n H' Y / f_0'') \, d\eta$$

$$V_{nj} = (\tfrac{1}{2} B_j \gamma_j) \int_0^\infty (M_n M_j H' / f_0'') \, d\eta$$

$$Q_n = \int_0^\infty (M_n H / f_0'')' Y \, d\eta$$

$$R_{nj} = (\tfrac{1}{2} B_j \gamma_j) \int_0^\infty (M_n H / f_0'')' M_j \, d\eta$$

where $H(\eta)$ and $Y(\eta)$ denote functions depending on the eigenfunctions N_j and M_j, respectively, namely

$$H \equiv f_2 + \sum_{j=1}^{\infty} \hat{A}_n N_j$$

$$Y \equiv (1 - f_0') - \sum_{j=1}^{\infty} B_j M_j.$$

The right-hand side of equation (55) may be expressed in terms of χ and $\hat{g}$ so that $(1 - g_c)$ is a factor. As a result when equations (58) and (59) are substituted into the left-hand side of equation (55), g_c disappears as a parameter and there results an integral equation for $\hat{g}$ in the range $0 \leqslant \chi \lesssim \frac{1}{2}$. An iteration scheme for solving this equation is readily established since it is of the form of an inhomogeneous Volterra equation. In evaluating the integrals appearing in equations (58) and (59) it is found convenient to use the idea behind Filton's method for evaluating Fourier trigonometric coefficients of high order and to approximate only $\hat{g}(\xi)$, e.g. as is done here by a piecewise trapezoidal rule, and to evaluate analytically the resultant integrals. To illustrate suppose we wish to evaluate at a value of $\chi = N\Delta$ the integral

$$I \equiv \chi^{-3\gamma_j} \int_0^\chi \xi^{3\gamma_j - 1} \, \hat{g}(\xi) \, d\xi,$$

Δ is a constant interval into which the range of χ is divided. Approximate $\hat{g}(\xi)$ in the interval $(m-1)\Delta \leqslant \xi \leqslant m\Delta$ by

$$\hat{g}(\xi) \simeq \hat{g}_{m-1} + (\hat{g}_m - \hat{g}_{m-1})[\xi - (m-1)\Delta]/\Delta. \tag{60}$$

Then there is found

$$I \simeq N^{-3\gamma_j} \sum_{m=1}^{\infty} \left\{ \left[\frac{m\hat{g}_{m-1} - (m-1)\hat{g}_m}{3\gamma_j} \right] [m^{3\gamma_j} - (m-1)^{3\gamma_j}] + \left[\frac{\hat{g}_m - \hat{g}_{m-1}}{3\gamma_j + 1} \right] [m^{3\gamma_j + 1} - (m-1)^{3\gamma_j + 1}] \right\} \tag{61}$$

which is independent of Δ. The result given by equation (61) is far more accurate than the usual integration formulas applied to the entire integrand.

The iteration procedure for the solution of the integral equation may be applied first with only equation (58) considered. After convergence is obtained, this approximation to $\hat{g}$ is then used as a first approximation with both equations (58) and (59) included. It is found that final convergence to within one per cent over the entire range of χ of interest is obtained in four iterations if this procedure is followed.

Consider next the solution of the species equation. To simplify the notation with no loss in generality let $\varkappa$ be a particular mass fraction, e.g. the mass fraction of the injected gas, and let its value in the free stream be $\varkappa_e$. Take the point of view that $\varkappa_w$ is known but anticipate the result that $\varkappa_w(0) = \varkappa_e$. Then following through the analysis of the energy equation we obtain for the first- and second-order solution

$$\varkappa(s, \eta) = \varkappa_e + \int_0^s \bar{g}(s, \eta; \xi)\,(\mathrm{d}\varkappa_w/\mathrm{d}\xi)\,\mathrm{d}\xi + \int_0^\infty \int_0^s G_2(s, \eta; s_0, \eta_0) J(s_0, \eta_0)\,\mathrm{d}s_0\,\mathrm{d}\eta_0 = \varkappa^{(1)}(s, \eta) + \varkappa^{(2)}(s, \eta) \quad (62)$$

where $\bar{g}$, G_2, and J are as given previously and where $\varkappa_w(0) = \varkappa_e$.

Now the boundary condition determining $\varkappa_w$ expresses the fact that for steady flow there is no net flux of gas from the external stream into the porous surface; it is in terms of $\hat{s}$ given by

$$\varkappa_\eta(\hat{s}, 0) = \tfrac{4}{3}\hat{s}^{\frac{1}{3}}\,(\varkappa_w - \varkappa_c) \quad (63)$$

where $\varkappa_c$ is the concentration of $\varkappa$ interior to the porous surface. It is clear from a comparison of equations (62) and (63) with equations (55), (58) and (59) that the solution for $\hat{g}(\hat{s})$ yields not only the distribution of g_w but of $\varkappa_w$ as well; explicitly we have

$$\frac{\varkappa_e - \varkappa_w}{\varkappa_e - \varkappa_c} = \frac{1 - g_w}{1 - g_c} = \hat{g}. \quad (64)$$

The results of our numerical analysis for the first and second approximations to $\hat{g}$ are shown in Fig. 5 where we have used 10 terms in the eigenfunctions N_n and M_n. Also shown there for comparison purposes is the result which is given by [13] and which is presumably more accurate in that five terms in the expansion for χ are included. The satisfactory agreement will be noted. It is perhaps of interest to compare the results in Fig. 5 with those obtained from the higher order calculation for the two-dimensional case by Libby and Chen [8]; suppose we consider two stations, one on a cone at a distance x from the apex and one on a wedge an equal distance x from the edge with the same external flows and with the same values of the mass flux ratio $(\rho v)_w/\rho_e u_e$ on the two bodies. Then from equation (52) and the definition of ϵ given previously we conclude that at these two stations $(-\epsilon)/\chi = 3^{\frac{1}{2}}$; we thus find that $\hat{g}$ for the station on the cone given by the present analysis is considerably less than on the wedge which implies that the thermal protection afforded by the porous cooling is much less effective on the cone than on the wedge.

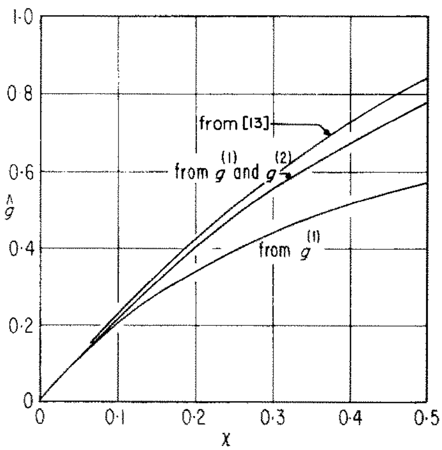

FIG. 5. Distribution of the surface parameter $\hat{g}$ with $\chi = \hat{s}^{\frac{1}{3}}$.

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Zusammenfassung—Eine Analyse wird durchgeführt für die Felder der Geschwindigkeit, der Energie und der Konzentration in laminaren Grenzschichten mit beliebig verteiltem Stoffübergang an der Oberfläche. Der Einfluss des Stofftransports wird dabei als Strömung behandelt. Die Einflüsse erster Ordnung sind einfach numerisch auszuwerten; Einflüsse höherer Ordnung lassen sich systematisch erhalten, aber erfordern bescheidenen Rechenaufwand zur numerischen Auswertung. Eine Anwendung erfolgte für die Grenzschicht mit gleichmässigem Stoffübergang und mit Energie- und Konzentrationsausgleich an der Oberfläche.

Аннотация—Проведен анализ полей скорости, энергии и концентрации в ламинарных пограничных слоях с произвольно распределенным переносом массы на поверхности. Анализ основывается на рассмотрении эффекта переноса массы как явления возмущения. Эффекты первого порядка численно рассчитываются довольно просто; эффекты более высокого порядка можно получить систематически, но их численный расчёт требует некоторых усилий. Анализ распространен на пограничный слой на конусе с однородным переносом массы в предположении баланса энергии и концентрации на поверхности.